EPFL– Fall 2025
Series 13

Differential Geometry I: Curves and surfaces

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13.1 Recall that, in exercise 10.7 we constructed the catenoid as the surface of revolution of the catenary curve:

$$\alpha(t) = (t, \cosh(t)).$$

In particular, the catenoid surface can be parametrized as follows (as a surface of revolution around the Ox-axis):

$$\psi(s,\theta) = \big(\log(s+\sqrt{1+s^2}),\ \sqrt{1+s^2}\cos\theta,\ \sqrt{1+s^2}\sin\theta\big).$$

Compute the metric tensor, the second fundamental form and the Weingarten map of the catenoid. What is its Gaussian curvature? Show that it's also a minimal surface, i.e. that its mean curvature vanishes.

Remark. The catenoid is the surface of minimal area with boundary given by two rings of the same radius which are sufficiently close to each other (think of a surface made of soap bubble connecting the two rings). When the rings are sufficiently far apart, however, the surface with this boundary which is of minimal area is instead the union of the two disjoint flat disks bounded by the rings.

13.2 Let $S \subset \mathbb{R}^3$ be a C^2 surface, and denote by H and K its mean and Gaussian curvature respectively.

(a) Show that the principal curvatures are given by

$$k_1,\ k_2 \ =\ H \pm \sqrt{H^2 - K}.$$

(b) Show that if S is a *minimal* surface, then its Gaussian curvature satisfies $K \leqslant 0$.

13.3 Show that the Gaussian curvature and the mean curvature can be expressed in terms of the coefficients (g_{ij}) and (h_{ij}) of the first and second fundamental forms by

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, \qquad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\,(g_{11}g_{22} - g_{12}^2)}.$$

13.4 Let $\psi_1 : \Omega \to S \subset \mathbb{R}^3$ be a surface of class C^2 and let $\lambda > 0$. Define $\psi_2 = \lambda\psi_1 : \Omega \to \lambda S \subset \mathbb{R}^3$, the surface obtained by applying a homothety of ratio λ. What is the relation between the Gaussian curvature $K_1(u,v)$ at a point $p = \psi_1(u,v) \in S$ and the Gaussian curvature $K_2(u,v)$ at the point $q = \lambda p = \psi_2(u,v) \in \lambda S$?

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13.5 A regular C^2 curve γ on a surface S is a *line of curvature* if its normal curvature is everywhere a principal curvature. Show that γ is a line of curvature if and only if its geodesic torsion is zero.

13.6 Prove that the catenoid (see Ex. 13.1) and the helicoid (see Ex. 10.6) are locally isometric (Hint: Consider the expression of the metric tensor in the parametrizations we have already considered). Are they globally isometric?

Remark. The fact that the catenoid and the helicoid are locally isometric implies that these two surfaces have the same Gaussian curvature at the corresponding points matched by the local isometry, as a corollary of the *Theorema Egregium.* If you want, you can try to confirm this by computation.

13.6 Let $S \subset \mathbb{R}^3$ be a surface of class C^2 and $\gamma : I \to S$ be a C^2 regular curve. We will say that γ is an *asymptotic* curve of S if the curvature vector $K_\gamma(t)$ of γ is tangent to S for all $t \in I$.

(a) Show that the following conditions are equivalent (for the last one, you can assume in addition that γ is a biregular curve):

1. γ is an asymptotic curve of S,
2. The normal curvature k_n of γ vanishes everywhere,
3. $h(\dot{\gamma}, \dot{\gamma}) = 0$ everywhere on γ,
4. The binormal of γ is parallel to the normal n of the surface.

(b) Let $\gamma : I \to S$ be an asymptotic curve of S. Prove that, for any $t \in I$, the Gauss curvature $K(\gamma(t))$ of S at $\gamma(t)$ satisfies $K(\gamma(t)) \leqslant 0$.

(c) Prove that, for a ruled surface, the line segments (corresponding to the ruling) are asymptotic curves (you don't really have to do a computation).

13.7 Let $\gamma : I \to \mathbb{R}^3$ be a C^3 biregular curve. Assume that $\|\dot{\gamma}\|$ is constant.

(a) Prove that γ is a geodesic of the ruled surface

$$S: \quad \psi(u, v) = \gamma(u) + v\, B_\gamma(u),$$

where $B_\gamma(u)$ is the binormal vector of γ.

(b) Prove that γ is an asymptotic of the ruled surface

$$S: \quad \psi(u, v) = \gamma(u) + v\, N_\gamma(u),$$

where $N_\gamma(u)$ is the principal normal of γ.

13.8 **(a)** First, let us recall some notions from linear algebra: For any symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$, we say that $A \geqslant 0$ if, for every $v \in \mathbb{R}^n$, we have

$$v^T A v \geqslant 0.$$

We also say that $A \geqslant B$ iff $A - B \geqslant 0$. Prove that, if $A \geqslant B$, then $\mathrm{Tr}(A) \geqslant \mathrm{Tr}(B)$.

(b) Let $\Omega \subseteq \mathbb{R}^2$ be a domain containing 0 and let $f_1, f_2 : \Omega \to \mathbb{R}$ be a pair of C^2 functions satisfying:

$$f_1(0) = f_2(0) = 0, \quad \nabla f_1(0) = \nabla f_2(0) = 0$$

and

$$f_1(x, y) \leqslant f_2(x, y) \quad \text{for all } (x, y) \in \Omega.$$

Let us denote by $S_1, S_2 \subset \mathbb{R}^3$ the graphs $\{z = f_1(x, y)\}$ and $\{z = f_2(x, y)\}$, respectively; note that S_1 and S_2 "touch" at 0. Let us fix the coorientations n_1, n_2 of S_1, S_2 so that, at 0, we have $n_1 = n_2 = (0, 0, 1)$. Show that the mean curvatures H_1, H_2 satisfy

$$H_1(0) \leqslant H_2(0).$$

Hint: You might want to first express the second fundamental forms of S_i at 0 in terms of $\nabla^2 f_i(0)$.

(c*) Show that there is no *compact* minimal surface in $\mathbb{R}^3$.

Hint: Argue by contradiction; assuming that S is a compact minimal surface, consider the smallest sphere S_R containing S in its interior. Such a sphere must touch S at a point p; apply part (b) for the surfaces S and S_R around that point.